about 13′ nearer the normal of the face (1 0 0) than that of the blue, the mean direction of the first mean line being about 35° on the normal of 1 0 0, and 8° 40′ on that of $\bar{1}$ 0 1.

Crystals of potassium nitrate, containing less silver nitrate than the double salt, gave angles according somewhat closely with those of ordinary nitre; but the crystals did not give very good reflections.

The results obtained by Rose with sodium nitrate containing silver nitrate were confirmed so far as the crystals permitted of measurement.

Crystals of strontium nitrate containing silver nitrate gave excellent measurements, according with those of a cubo-octahedron.

VIII. "On certain Definite Integrals." By W. H. L. Russell, F.R.S. Received June 21, 1877.

The following paper is a continuation of two papers recently communicated to the Royal Society, and inserted in the 'Proceedings.'

$$(40.)\ \int_0^\pi d\theta \log_\epsilon\left(\epsilon^{2x\cos\theta}+2\epsilon^{x\cos\theta}\cos(x\sin\theta)+1\right)=2\pi\log_\epsilon 2.$$

$$(41.)\ \int_0^{\frac{\pi}{2}} d\theta\, \epsilon^{2\cos^4\theta}\cos(\sin 2\theta\cos^2\theta+\sin\theta\cos\theta)=\frac{\pi}{2}\sqrt[4]{\epsilon^3}.$$

$$(42.)\ \int_0^\pi d\theta\, \epsilon^{\cos 2\theta+\cos\theta}\cos(\sin 2\theta+\sin\theta)=\pi.$$

$$(43.)\ \int_0^\pi d\theta\, \epsilon^{\frac{1-x^2}{1-2x\cos\theta+x^2}}\cos\frac{2x\sin\theta}{1-2x\cos\theta+x^2}=\pi\epsilon.$$

$$(44.)\ \int_0^\pi d\theta\,.\,\epsilon^{\frac{1-x\cos\theta}{1-2x\cos\theta+x^2}}\cos\frac{x\sin\theta}{1-2x\cos\theta+x^2}=\pi\epsilon.$$

$$(45.)\ \int_0^\pi \theta d\theta\,.\frac{\sin\theta(4+\cos^2\theta)}{(3+\sin^2\theta)^2}=\frac{\pi}{3}.$$

$$(46.)\ \int_0^\pi \frac{\theta\sin^3\theta d\theta}{1-x^2\cos^2\theta}=\frac{\pi(x^2-1)}{2x^3}\log\epsilon\frac{1+x}{1-x}+\frac{\pi}{x^2}.$$

$$(47.)\ \int_0^\pi \frac{\theta\sin^3\theta\, d\theta}{1+x^2\cos^2\theta}=\frac{\pi(x^2+1)}{x^3}\tan^{-1}x-\frac{\pi}{x^2}.$$

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Continuing the process, we find

(48.) $\int_0^\pi \frac{\theta \sin^5\theta d\theta}{1+x^2\cos^2\theta}$, and more generally (49.) $\int_0^\pi \frac{\theta \sin^{2n+1}\theta d\theta}{1+x^2\cos^2\theta}$.

$$(50.)\ \int_0^\infty \frac{d\theta \sin^2 a\theta}{\left((1+\mu+\mu^2)\cos a\theta + \mu\cos 3a\theta\right)(1+\theta^2)} = \frac{\pi}{2-2\mu}\cdot\frac{\epsilon^{2a}-1}{\epsilon^{2a}+1}\cdot\frac{\epsilon^a}{\epsilon^{2a}+\mu}.$$

$$(51.)\ \int_0^{\frac{\pi}{2}} d\theta\, \epsilon^{\cos^n\theta \cos n\theta} \cos(\cos^n\theta \sin n\theta) = \frac{\pi}{2}\cdot\sqrt[2^n]{\epsilon}.$$

$$(52.)\ \int_0^\pi \frac{d\theta \sin\frac{\theta}{2}\epsilon^{\cos\theta}\sin\left(\theta+\frac{\theta}{2}\right)}{1-2a\cos\theta+a^2} = \frac{\pi}{2}\frac{\epsilon^a}{1+a}.$$

$$(53.)\ \int_0^\pi \frac{d\theta(1+\mu\,\epsilon^{\cos\theta}\cos\sin\theta)}{(1-2a\cos\theta+a^2)(1+2\mu\,\epsilon^{\cos\theta}\cos\sin\theta+\mu^2\epsilon^{2\cos\theta})} = \frac{\pi}{1-a^2}\cdot\frac{1}{1+\mu\epsilon^a}.$$

$$(54.)\ \int_0^\infty \frac{d\theta\,(1+\mu\epsilon^{\cos\theta}\cos\sin\theta)}{(1+\theta^2)(1+2\mu\epsilon^{\cos\theta}\cos\sin\theta+\mu^2\epsilon^{2\cos\theta})} = \frac{\pi}{2}\cdot\frac{1}{1+\mu\epsilon^{\frac{1}{\epsilon}}}.$$

$$(55.)\ \int_0^\pi \frac{\epsilon^{\cos\theta}\sin(\sin\theta)\,d\theta\sin\theta}{(1-2a\cos\theta+\alpha^2)(1+2\mu\epsilon^{\cos\theta}\cos\sin\theta+\mu^2\epsilon^{2\cos\theta})} = \frac{\pi}{2a}\left\{\frac{\epsilon^\alpha}{1+\mu\epsilon^\alpha}-\frac{1}{1+\mu}\right\}$$

$$(56.)\ \int_0^\pi \frac{\log_\epsilon(1-2a\cos\theta+a^2)}{1+2x\cos\theta+x^2} = \frac{2\pi}{1-x^2}.\log_\epsilon(1+ax).$$

In like manner we may find the integral (57.) $\int_0^\infty \frac{d\theta}{(1+\theta^4)(1-2a\cos\theta+a^2)}$;

and more generally (58.) $\int_0^\infty \frac{d\theta}{(1+\theta^{2n})(1-2a\cos\theta+a^2)}$.

$$(59.)\ \int_0^\pi \frac{d\theta\,.\sin^2\theta}{(1-2a\cos\theta+a^2)(1+2x\cos\theta+x^2)^2} = \frac{\pi}{2(1-x^2)}\cdot\frac{1}{(1+ax)^2}.$$

$$(60.)\ \int_0^\infty \frac{\theta\sin\theta\,d\theta}{(1+\theta^2)(1+2x\cos\theta+x^2)^2} = \frac{\pi}{2(1-x^2)}\cdot\frac{\epsilon}{(\epsilon+x)^2}.$$

$$(61.)\ \int_0^\infty \frac{d\theta\sin\theta}{(\theta+\theta^3)(1+2x\cos\theta+x^2)^2} = \frac{\pi}{2}\frac{(\epsilon-1)(\epsilon-x^2)}{(1-x^2)(1+x)^2(\epsilon+x)^2}.$$

I have been asked to indicate the method of obtaining some of these integrals. This I now do.

(8.) is obtained by summing the series for $\frac{\sin x}{x}$ by means of the definite integral

$$\int_0^{\frac{\pi}{2}} \log_\epsilon \cos\theta \cos 2r\theta = (-1)^{r-1}.\frac{\pi}{4}.\frac{1}{r}.$$

(9.) by expanding $\sqrt{1+\frac{\alpha}{2}}$ in terms of α, adding the resulting series by means of the definite integral

$$\int_0^{\frac{\pi}{2}} d\theta \cos^n\theta \cos n\theta = \frac{\pi}{2}.\frac{1}{2^n},$$

and combining the definite integrals arising from the process, by the rule for the addition of binomial surds. To obtain (10.) we expand $\frac{1}{\epsilon^{\frac{\alpha}{2}}+1}$ in terms of α, and then sum the series by means of the integral

$$\int_0^{\frac{\pi}{2}} d\theta \cos^n\theta \cos n\theta = \frac{\pi}{2}.\frac{1}{2^n}.$$

(13.) is found in a similar manner. (14.) is obtained by using the integral $\int_0^\infty \frac{d\theta \cos a\theta}{1+\theta^2}$, and (16.) by means of the integral which we used to obtain (8). (19.) is found by summing up the series for $\int \frac{dx}{\epsilon^x+1}$ by means of the definite integral $\int_0^\pi \theta \sin n\theta = (-1)^{n+1}\frac{\pi}{n}$. (40.) is derived from (19.) by integrating by parts; but this integral can also be obtained by expansion. (21.) and (22.) are obtained from the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^{n-1}\theta\, d\theta \cos\{c\tan\theta+(n-1)\theta\}\, d\theta = \frac{\pi}{2^n}\epsilon^{-c}.$$

(33.) is found by using the integral

$$\int_0^{\frac{\pi}{2}} d\theta\, \theta \cos^{n-1}\theta \sin(n+1)\theta = \frac{\pi}{2^{n+1}}.\frac{1}{n}.$$

(34.), (35.), and (50.) arise from the integrals

$$\int_0^\omega \frac{\cos 2ra\theta\,.\,\theta\,d\theta}{\sin a\theta\,(1+\theta^2)} \text{ and } \int_0^\omega \frac{\cos 2ra\theta\,.\,d\theta}{\cos a\theta\,(1+\theta^2)^2}.$$

(53.), (54.), and (55.) are obtained by using series analogous to

$$\epsilon^{\cos\theta}\cos\sin\theta+\mu\epsilon^{2\cos\theta}\cos(2\sin\theta)+\mu^2\epsilon^{3\cos\theta}\cos(3\sin\theta)+\ldots,$$

and (59.), (60.), and (61.) are found by the series

$$x\sin\theta-2x^2\sin 2\theta+3x^3\sin 3\theta-\ldots=\frac{x\sin\theta\,(1-x^2)}{(1+2x\cos\theta+x^2)^2}.$$

The series employed are considered as convergent, according to the rule given by Cauchy. On account of the great importance of the subject of convergent series, I have thought that mathematicians would be interested to see how the results of Cauchy are confirmed by a method given by me in the 'Proceedings of the Royal Society' for 1872, vol. xxi. page 20, and which was afterwards discovered independently by Sir W. Thomson. Let $u_0+u_1x+u_2x^2+\ldots+u_nx^n+\ldots$ be any series, then I call $\frac{u_{n+1}}{u_n}$ when n increases without limit the ultimate ratio of the series, and denote it by ρ. Then if x is less than $\frac{1}{\rho}$, the series will be convergent.

Now consider the function $\frac{1}{\epsilon^x+1}$ which we have employed in these investigations.

Since $\frac{1}{\epsilon^x+1}+\frac{1}{\epsilon^{-x}+1}=1$, the series comprising $\frac{1}{\epsilon^x+1}$ can contain no even powers, and therefore we may assume

$$\frac{1}{\epsilon^x+1}=u_0+u_1x+u_3x^3+u_5x^5+\ldots$$

Expanding ϵ^x+1 in terms of x, and multiplying the series together, we obtain

$$2u_{2n+1}+\frac{u_{2n-1}}{1\,.\,2}+\frac{u_{2n-3}}{1\,.\,2\,.\,3\,.\,4}+\ldots=0;$$

$$\text{or } 1+1+\frac{u_{2n-1}}{u_{2n+1}}\,.\,\frac{1}{1\,.\,2}+\frac{u_{2n-3}}{u_{2n-1}}\,.\,\frac{u_{2n-1}}{u_{2n+1}}\,.\,\frac{1}{1\,.\,2\,.\,3\,.\,4}+\ldots=0.$$

Let n increase without limit, and then

$$\frac{u_{2n+1}}{u_{2n-1}}=\frac{u_{2n-1}}{u_{2n-3}}=\ldots=\rho,$$

whence $1 + \cos \frac{i}{\rho^{\frac{1}{2}}} = 0$, which gives

$$\cdot\rho = -\frac{1}{\pi^2}, \text{ or } -\frac{1}{9\pi^2}, \text{ or } -\frac{1}{25\pi^2} \cdots;$$

that is to say, the series will be convergent if $\frac{x^2}{\pi^2}, \frac{x^2}{9\pi^2}, \frac{x^2}{25\pi^2} \ldots$ are all less than unity—that is, if x is less than π; which accords with the result given by Cauchy.

Let us next consider the series

$$\epsilon^{x^2+x} = u_0 + u_1 x + u_2 x^2 + \ldots,$$

we easily obtain by differentiation the following equation for determining u_n,

$$(n+1)\, u_{n+1} = u_n + 2u_{n-1},$$

which may be written

$$(n+1)\frac{u_{n+1}}{u_n} \cdot \frac{u_n}{u_{n-1}} = \frac{u_n}{u_{n-1}} + 2.$$

As n increases without limit, $\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} = \rho$, and the equation becomes $\rho^2 = 0$, or $\rho = 0$, whence the series is convergent however great x may become.

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Transactions.

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